III. Results and Discussion

To the authors' knowledge, there exist at least two other approaches to solve boundary-layer equations resulting in block tri-diagonal matrix. One is that by Keller and Cebeci,[14-16] and the other is that by Libby and Wu.[17-18] Liu and Davy have tried them both.[11] Both approaches give accurate results for low blowing. However, neither of them prove to be successful for $f_w < -4$ even with very good initial guesses. In an unpublished note, the first author and M.J. Green of Ames Research Center calculate the condition numbers resulting from both matrices. We are able to prove that the condition numbers increase with $|f_w|$ and therefore the factorization procedure is unstable.

The numerical method we have discussed in the previous sections was coded in FORTRAN and calculations have beem made for the case of very large blowing $f_w = -30$. The results obtained from IBM 360-67 computer are shown in Figs. 1 and 2. The initial guesses are shown by the dashed lines. The inner portion of the initial guess is that of Libby's zeroth order inner solution.[12] The node points are chosen so that Eq. (31) is satisfied. The convergence is reached in 7th iteration with an error less than 1% for all the node points. Each iteration takes about 2 sec. Several cases with different β, f_w, and S_w have been tried. They all converged in less than 10 iterations and over all computing time is little and independent of blowing parameter.

The same scheme has been tried on a non-similar turbulent boundary layer with mass and heat transfer over a rough surface with two-equation model for closure. No difficulty has been encountered yet. We shall report the results elsewhere.

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Stresses and Displacements in Rotating Anisotropic Disks with Variable Densities

C.I. Chang*

Westinghouse Electric Corporation, Madison, Pa.

THE problem of rotating disks was first treated in the early nineteenth century. Solutions of the isotropic disks, including variable thickness, variable density, and other cases, can be found in most of the standard elasticity textbooks. Therefore, rotating disks were considered to be one of the exhausted subjects in the field of solid mechanics. Recently, new interest has been generated to reinvestigate this centuries-old problem. This new interest is the result of the development of composite materials and their applications.

Composite materials are characterized by high strength-to-weight ratios, heterogeneity and anisotropic characteristics. Because of the heterogeneity and anisotropic characteristics, it makes the effort in predicting the material response under multiaxial states of stress more difficult.[1] Recently, the rotating disk technique has been proved to be a simple and reliable means of generating a biaxial state of stress when the loads cannot be directly applied to the material under investigation. In this case, an analytical solution of this problem is needed to interpret the experimental results generated in the laboratory. Recently, because of the steady increase in the use of energy and the impact of that use on the environment, a flywheel made of composite materials has been proved to be an efficient means of storing energy.[2,3] Likewise, analytical solution of rotating disks made of composite materials is essential in design and analysis.

Tang[4] and Murthy and Sherbourne[5] have treated the cases of rotating polar orthotropic disks of uniform and nonuniform thickness, respectively. Reddy and Srinath[6] further investigated the effect of material density on the stresses and displacements of a rotating polar orthotropic circular disk. In recent investigations, closed-form solutions of rotating circular and elliptical disks made of general orthotropic materials have been developed by the author.[7,8]

Density variation in the material is, however, inevitable, and it has been shown that density variation has significant effects on the stresses and displacements of a rotating polar orthotropic disk. It is the purpose of this note to present a closed-form solution of a rotating orthotropic circular disk of which a radial density variation is assumed. A semi-inverse technique is employed, and the material is assumed to be linear elastic.

Analysis

For an orthotropic cylinder, if the axial displacement vanishes and the displacement components on any plane per-

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*Senior Engineer, Reactor Internal Stress Analysis, Advanced Reactors Division.

pendicular to the generator axis are only functions of the cartesian coordinates x, y within the plane, the stress functions $F(x, y)$ and $\psi(x, y)$ and boundary conditions on the lateral surface can be found in Ref. 9. If the system axes are selected to coincide with the principal material axes, only nine independent material constants remain. With this simplification governing differential equations reduce into forms

$$\beta_{22}\frac{\partial^4 F}{\partial x^4}+(2\beta_{12}+\beta_{66})\frac{\partial^4 F}{\partial x^2\partial y^2}+\beta_{11}\frac{\partial^4 F}{\partial y^4}=$$

$$-(\beta_{12}+\beta_{22})\frac{\partial^2 \bar{U}}{\partial x^2}-(\beta_{11}+\beta_{12})\frac{\partial^2 \bar{U}}{\partial y^2} \tag{1a}$$

$$\beta_{44}\frac{\partial^2 \psi}{\partial x^2}+\beta_{55}\frac{\partial^2 \psi}{\partial y^2}=-2\bar{\theta}+Aa_{34}-Ba_{35} \tag{1b}$$

where $\bar{\theta}$, A, B are constants which can be determined from the torque and bending moments applied on the ends, and a_{ij} and β_{ij} elastic constants.

Considering the problem of a rotating cylinder with a radial density variation described by the equation

$$\rho=\rho_0(1+r/a)^m \tag{2}$$

where ρ_0 is the density at the center of the disk, a is the radius, and m is the constant characterizing the density variation within the disk.

The boundary conditions become

$$\frac{\partial F}{\partial x}=\int_0^s(\tfrac{1}{2})\rho_0\left(1+\frac{r}{a}\right)^m\omega^2(x^2+y^2)\frac{dx}{ds}\,ds \tag{3a}$$

$$\frac{\partial F}{\partial y}=\int_0^s(\tfrac{1}{2})\rho_0\left(1+\frac{r}{a}\right)^m\omega^2(x^2+y^2)\frac{dy}{ds}\,ds \tag{3b}$$

$$\psi=0 \tag{3c}$$

where ω is the angular velocity.

It can be verified that the stress functions F and ψ which satisfy both boundary conditions (3) and differential equations (2) are

$$F=\frac{\rho_0(1+r/a)^m\omega^2}{8}K(x^2+y^2-a^2)$$

$$+2^{m-2}\rho_0\omega^2a^2\left[(x-a)^2+y^2\right] \tag{4a}$$

$$\psi=0 \tag{4b}$$

where

$$K=\frac{2\beta_{12}+\beta_{11}+\beta_{22}}{3(\beta_{11}+\beta_{22})+(2\beta_{12}+\beta_{66})} \tag{5}$$

The stress components of a rotating cylinder are obtained from the calculated potential F and ψ and they are

$$\tau_x=\tfrac{1}{2}\rho\omega^2\left[K(x^2+3y^2-a^2)-(x^2+y^2)\right]+2^{m-1}\rho_0\omega^2a^2 \tag{6a}$$

$$\tau_y=\tfrac{1}{2}\rho\omega^2\left[K(3x^2+y^2-a^2)-(x^2+y^2)\right]+2^{m-1}\rho_0\omega a^2 \tag{6b}$$

$$\tau_{xy}=-\rho\omega^2Kxy \tag{6c}$$

$$\tau_{xz}=\tau_{yz}=0 \tag{6d}$$

$$\tau_z=-1/a_{33}(a_{13}\tau_x+a_{23}\tau_y) \tag{6e}$$

For the case of a thin rotating disk, this becomes a plane stress solution, and the average stress components through the thickness of the disk are obtained by replacing β_{ij} by a_{ij} and setting τ_z equal to zero. If the results are transformed into polar coordinates, the radial, circumferential, and shear stress components are

$$\sigma_r=\tfrac{1}{2}\rho_0\left[1+\frac{r}{a}\right]^m\omega^2\left[K(r^2-a^2)-r^2\right]+2^{m-1}\rho_0\omega^2a^2 \tag{7a}$$

$$\sigma_\theta=\tfrac{1}{2}\rho_0\left[1+\frac{r}{a}\right]^m\omega^2\left[K(3r^2-a^2)-r^2\right]+2^{m-1}\rho_0\omega^2a^2 \tag{7b}$$

$$\tau_{r\theta}=0 \tag{7c}$$

It is interesting to note that, similar to the case of the isotropic disk, the stress distributions of a variable-density orthotropic circular disk are functions of the radial coordinate only. Shear stress is identically equal to zero.

Stress distributions for various density variations in the disk are plotted in Figs. 1 and 2. The curve representing the constant-density disk is labeled by $m=0$. It is amazing to note that a linear radially increasing density distribution ($m=1$) doubles the maximum stress at the center of the disk. On the other hand, a linear radially decreasing density distribution ($m=-1$) reduces the maximum stress at the center of the disk by nearly 60%.

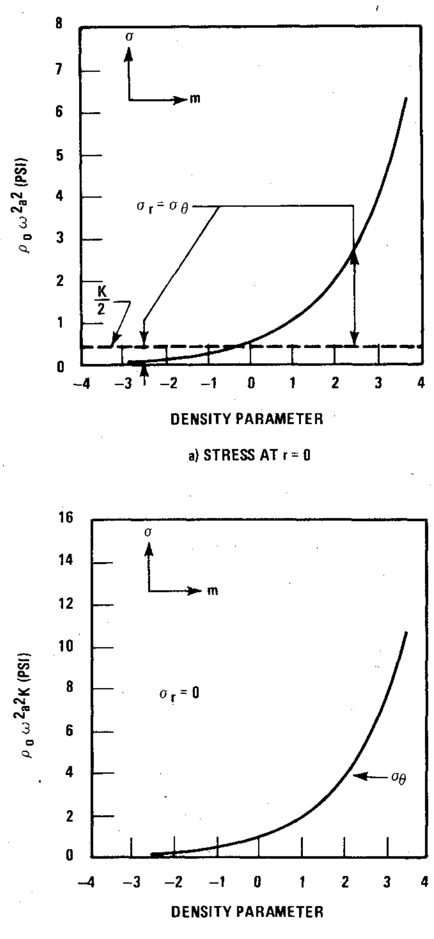

Fig. 1 Stresses at center and periphery of rotating orthotropic disks with various density distributions.

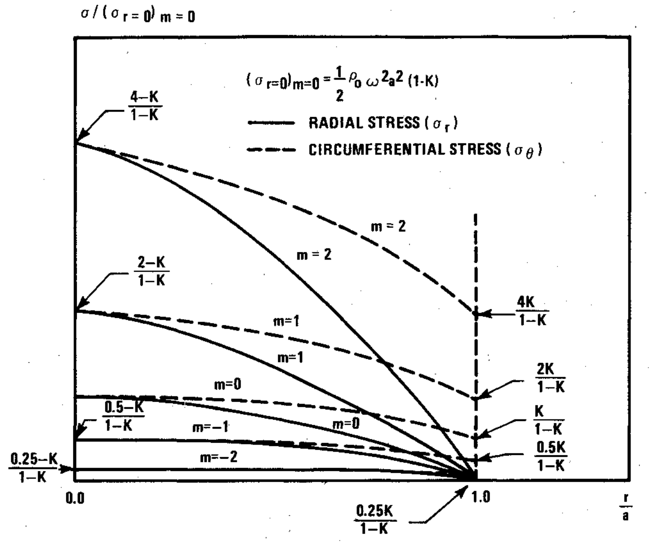

Fig. 2 Stress distrubition in rotating orthotropic disks with various densities.

When m is set equal to zero, Eq. (7) becomes

$$\sigma_r = \tfrac{1}{2}\rho_0\omega^2 a^2 (1-K)(a^2-r^2) \tag{8a}$$

$$\sigma_\theta = \tfrac{1}{2}\rho_0\omega^2 a^2 \left[K(3r^2-a^2) + (a^2-r^2) \right] \tag{8b}$$

which is the solution of a rotating orthotropic disk with constant density.[7,8]

Substituting the stress components in Eq. (7) into stress-strain relationships and integrating the results, the displacements along the principal material direction x and y are

$$U = \frac{a_{11}\rho\omega^2 x}{2}\left\{K\left(\frac{x^2}{3}+3y^2-a^2\right)-\left(\frac{x^2}{3}+y^2\right)\right\} + 2^{m-1}a_{11}\rho_0\omega^2a^2x + \frac{a_{12}\rho\omega^2 x}{2}\left\{K(x^2+y^2-a^2) - \left(\frac{x^2}{3}+y^2\right)\right\} + 2^{m-1}a_{12}\rho_0\omega^2a^2x \tag{9a}$$

$$V = \frac{a_{21}\rho\omega^2 y}{2}\left\{K(x^2+y^2-a^2)-\left(x^2+\frac{y^2}{3}\right)\right\} + 2^{m-1}a_{21}\rho_0\omega^2a^2y + \frac{a_{22}\rho\omega^2 y}{2}\left\{K\left(3x^2+\frac{y^2}{3}-a^2\right) - \left(x^2+\frac{y^2}{3}\right)\right\} + 2^{m-1}a_{22}\rho_0\omega^2a^2y \tag{9b}$$

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Convergence Control in Differential Dynamic Programing Applied to Air-to-Air Combat

Bernt S. A. Jarmark*
Royal Institute of Technology, Stockholm and SAAB-SCANIA, Linköping, Sweden

ONE particular difficulty when solving differential game problems numerically is convergence control, i.e., the measures taken between iterations to ensure the stability of the approximation and to speed up the convergence. Most often when solving practical optimization problems one of the following methods will be used: neighboring extremal methods, gradient method, quasilinearization method, or, as in the present case, differential dynamic programing (DDP).

This Note presents a new convergence control technique for DDP, which is applicable to the computation of optimal trajectories for aircraft. When evaluating such trajectories, the author first tried some algorithms of neighboring extremals. The usual difficulties due to the missing boundary values when solving TPBVP's were encountered. In particular, the convergence domain of the initial values is too small. On the other hand DDP very rapidly resulted in an acceptable solution. In DDP the initial value problem corresponds to the situation when a nominal trajectory is far away from the optimal. In this case the established way is to use the step-size-method of Jacobson-Mayne.[1] A new method—an alternative to the step-size-method—developed by the author has several advantages, particularly considering differential game problems. For a test, the new method also was applied to an example from Ref. 1. (pp. 35-38)—the Rayleigh equation. It provided good convergence.

Concerning optimal trajectories for aircraft the state vector turns out to have a dimension larger than five. This gives a strong increase in the number of equations when using a second-order expansion. In this type of problem the exact optimal trajectory need not be reached because, in practice, it cannot be implemented. Therefore, in most cases, a first-order expansion has been deemed satisfactory when using DDP with the new method, although it would work for the second-order.

The purpose of this paper is to demonstrate: 1) A method which could be an alternative to the step-size method.[1] 2) An extension of the application of DDP to differential game, air-to-air combat problems. This is done by comparison to Ref. 5 in which the gradient method and the neighboring method have been applied.

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*Research Engineer, Division of Automatic Control, Royal Institute of Technology and Aerospace Division of SAAB-SCANIA.